3. Propositional Logic – Exercises

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Propositional Logic - Practicing Natural Deduction

Natural Deduction Rules

Last RAMDE's class...

On the last class, you were introduced to Propositional Logic:

- its syntax and semantics
- normal forms: negative, disjunctive, and conjunctive
- rules for natural deduction

During this class...

You will be exposed to the practice of construction proofs about Propositional Logic's formulae using Natural Deduction

Warning: Becoming comfortable with this type of mathematics is not an easy task! Bare with me and be pattient! Train a lot by doing the exercises at home once again, to start solidifying the types of proof patterns that naturally will appear...

Recalling the rules of introduction and elimination: conjunction

Introduction

If we know that both φ and ψ is hold, then so does their conjunction.

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \wedge\mathbf{I}$$

Elimination

If know that $\varphi \wedge \psi$, then we can conclude that either of them also holds in isolation.

$$\frac{\varphi \wedge \psi}{\varphi} \wedge\mathbf{E}_l \qquad\qquad \frac{\varphi \wedge \psi}{\psi} \wedge\mathbf{E}_r$$

Quick exercise

Exercise

Prove that if $\varphi \wedge \psi$ holds, then $\psi \wedge \varphi$ also holds. That is $\varphi \wedge \psi \vdash \psi \wedge \varphi$

1	$\varphi \wedge \psi$	
2	φ	$\wedge\mathbf{E}_l(1)$
3	ψ	$\wedge\mathbf{E}_r(1)$
4	$\psi \wedge \varphi$	$\wedge\mathbf{I}(2,3)$

Recalling the rules of introduction and elimination: disjunction

Introduction

We can construct a new disjunction $\varphi \vee \psi$ if we know that either φ or ψ hold.

$$\frac{\varphi}{\varphi \vee \psi} \vee \mathbf{I}_l \qquad\qquad \frac{\psi}{\varphi \vee \psi} \vee \mathbf{I}_r$$

Elimination

The elimination, in this case, assumes the form of introducing a new formula θ in case we can derive θ from both φ and ψ, and we know that $\varphi \vee \psi$ holds.

$$\frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \theta\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \theta\end{matrix}}{\theta} \vee \mathbf{E}$$

Quick exercise

Exercise

Prove that if $(\varphi \vee \psi) \wedge \theta$ holds, then $(\varphi \wedge \theta) \vee (\psi \wedge \theta)$ also holds.

1	$(\varphi \vee \psi) \wedge \theta$	
2	$\varphi \vee \psi$	$\wedge\mathbf{E}_l(1)$
3	θ	$\wedge\mathbf{E}_r(1)$
4	φ	
5	$\varphi \wedge \theta$	$\wedge\mathbf{I}(4, 3)$
6	$(\varphi \wedge \theta) \vee (\psi \wedge \theta)$	$\vee\mathbf{I}_l(5)$
	$\vdots$	
7	ψ	
8	$\psi \wedge \theta$	$\wedge\mathbf{I}(8, 3)$
9	$(\varphi \wedge \theta) \vee (\psi \wedge \theta)$	$\vee\mathbf{I}_r(8)$
10	$(\varphi \wedge \theta) \vee (\psi \wedge \theta)$	$\vee\mathbf{E}(2, 4–6, 7–9)$

Recalling the rules of introduction and elimination: Negation

Introduction

If we can derive false from φ, then we can conclude that φ does not hold, that is, its negation $\neg\varphi$ holds.

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots\\ \bot\end{array}}{\neg\varphi}\ \neg\mathbf{I}$$

Elimination

If we know that $\neg\varphi$ is false, then ew can conclude that φ holds.

$$\dfrac{\neg\neg\varphi}{\varphi}\ \neg\mathbf{E}$$

Recalling the rules of introduction and elimination: False

Introduction

If we assume φ and, still, we are able to derive $\neg\varphi$, then we can conclude false. In fact, we found a contradiction!

$$\dfrac{\begin{array}{c}\varphi \\ \vdots \\ \neg\varphi\end{array}}{\bot}\ \bot\mathbf{I}$$

Elimination

From false, ew can conclude anything!

$$\dfrac{\bot}{\varphi}\ \bot\mathbf{E}$$

Quick exercise

Exercise

Prove that if $\neg(\varphi \vee \psi)$ holds, then $\neg\varphi \wedge \neg\psi$ also holds.

1	$\neg(\varphi \vee \psi)$	
2	φ	$\wedge\mathbf{E}_l(1)$
3	$\varphi \vee \psi$	$\vee\mathbf{I}_l(2)$
4	$\bot$	$\bot\mathbf{I}(1,3)$
5	$\neg\varphi$	$\neg\mathbf{I}(2\text{–}4)$
6	ψ	
7	$\varphi \vee \psi$	$\vee\mathbf{I}_r(6)$
8	$\bot$	$\bot\mathbf{I}(1,7)$
9	$\neg\psi$	$\neg\mathbf{I}(6\text{–}8)$
10	$\neg\varphi \wedge \neg\psi$	$\wedge\mathbf{I}(5,8)$

Natural Deduction Rules - Implication

Introduction of implication

If we assume φ and we can derive ψ from it, then we can conclude that $\varphi \to \psi$.

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \to\mathbf{I}$$

Elimination of implementation

From false, we can conclude whatever we want.

$$\dfrac{\varphi \to \psi \qquad \varphi}{\psi} \to\mathbf{E}$$

Quick exercise

Exercise

Prove that if $(\varphi \vee \psi) \to \theta$ and φ hold, then $\psi \to \theta$ also holds.

1	$(\varphi \vee \psi) \to \theta$	
2	φ	
3	$\quad \psi$	
4	$\quad \varphi \vee \psi$	$\vee\mathbf{I}_l(2)$
5	$\quad \theta$	$\to\mathbf{E}(1, 4)$
6	$\psi \to \theta$	$\to\mathbf{I}(3–5)$

Natural Deduction Rules - Derived rules

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}\ \textbf{MT}$$

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg\textbf{I}$$

$$\begin{array}{c}[\neg\varphi] \\ \vdots \\ \dfrac{\bot}{\varphi}\ \textbf{RA}\end{array}$$

$$\frac{}{\varphi \vee \neg\varphi}\ \textbf{ET}$$

Lets prove the derived rules?

Exercise

Prove that $\varphi \to \psi, \neg\psi \vdash \neg\varphi$

1	$\varphi \to \psi$	
2	$\neg\psi$	
3	$\quad \varphi$	
4	$\quad \psi$	$\to$**E**$(1, 3)$
5	$\quad \bot$	$\bot$**I**$(3, 4)$
6	$\neg\varphi$	$\neg$**I**$(3\text{–}5)$

Lets prove the derived rules?

Exercise

Prove that $\varphi \vdash \neg\neg\varphi$

1	φ	
2	$\quad \neg\varphi$	
3	$\quad F$	$\bot\mathbf{I}(1,2)$
4	$\neg\neg\varphi$	$\neg\mathbf{I}(2\text{–}3)$

Lets prove the derived rules?

Exercise

Prove that $\neg\varphi \to F \vdash \varphi$

1	$\neg\varphi \to F$	
2	$\quad \neg\varphi$	
3	$\quad F$	$\perp\mathbf{I}(1,2)$
4	$\neg\neg\varphi$	$\neg\mathbf{I}(2\text{–}3)$
5	φ	$\neg\mathbf{E}(4)$

Lets prove the derived rules?

Exercise

Whatever φ we have $\vdash \varphi \vee \neg\varphi$

1	$\neg(\varphi \vee \neg\varphi)$	
2	φ	
3	$\varphi \vee \neg\varphi$	$\vee\mathbf{I}_r(2)$
4	$\bot$	$\bot\mathbf{I}(1,3)$
5	$\neg\varphi$	$\neg\mathbf{I}(2\text{—}4)$
6	$\varphi \vee \neg\varphi$	$\vee\mathbf{I}_l(5)$
	$\vdots$	
7	$\bot$	$\bot\mathbf{I}(1,6)$
8	$\neg\neg(\varphi \vee \neg\varphi)$	$\neg\mathbf{I}(1\text{—}7)$
9	$\varphi \vee \neg\varphi$	$\neg\mathbf{E}(8)$

Lets continue with more exercises

Exercise

Build the derivations for each of the statements below:

- $\vdash (\varphi \wedge \psi) \rightarrow \psi$
- $\vdash \varphi \rightarrow (\varphi \vee \psi)$
- $\vdash (\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$
- $\theta \rightarrow (\varphi \rightarrow \psi), \neg\psi, \theta \vdash \neg\varphi$
- $\theta, \neg\varphi \vdash \neg(\theta \rightarrow \varphi)$
- $(\psi \wedge \theta) \rightarrow \neg\delta, \varphi \rightarrow \delta, \theta, \varphi \vdash \neg\psi$
- $(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$

Solutions for each of the statements are given in the slides that follow...

Solution for $\vdash (\varphi \land \psi) \to \psi$

1	$\varphi \land \psi$	
2	ψ	$\land \mathbf{E}_l(1)$
3	$(\varphi \land \psi) \to \psi$	$\to \mathbf{I}(1\text{–}2)$

Solution for $\vdash \varphi \to (\varphi \lor \psi)$

1	$\quad \mid \; \varphi$	
2	$\quad \mid \; \varphi \lor \psi$	$\lor \mathbf{I}_r(1)$
3	$\varphi \to (\varphi \lor \psi)$	$\to\mathbf{I}(1\text{–}2)$

Solution for $\vdash (\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$

1	$\varphi \vee \psi$	
2	φ	
3	$\psi \vee \varphi$	$\vee\mathbf{I}_r(2)$
4	ψ	
5	$\psi \vee \varphi$	$\vee\mathbf{I}_l(4)$
	$\vdots$	
6	$\psi \vee \varphi$	$\vee\mathbf{E}(1, 2{-}3, 4{-}5$
7	$(\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$	$\rightarrow\mathbf{I}(1{-}6)$

Solution for $\theta \to (\varphi \to \psi), \neg\psi, \theta \vdash \neg\varphi$

1	$\varphi \to \psi$	
2	$\neg\psi$	
3	θ	
4	$\quad \varphi$	
5	$\quad \psi$	$\to$**E**$(1, 4)$
6	$\quad \bot$	$\bot$**I**$(2, 5)$
7	$\neg\varphi$	$\neg$**I**$(4\text{–}6)$

Solution for $\theta, \neg\varphi \vdash \neg(\theta \to \varphi)$

1	θ	
2	$\neg\varphi$	
3	$\quad \theta \to \varphi$	
4	$\quad \varphi$	$\to$**E**$(1, 3)$
5	$\quad \bot$	$\bot$**I**$(2, 4)$
6	$\neg(\theta \to \varphi)$	$\neg$**I**$(3$–$5)$

Solution for $(\psi \wedge \theta) \to \neg\delta, \varphi \to \delta, \theta, \varphi \vdash \neg\psi$

1	$(\psi \wedge \theta) \to \neg\delta$	
2	$\varphi \to \delta$	
3	θ	
4	φ	
5	$\quad \psi$	
6	$\quad \psi \wedge \theta$	$\wedge$**I**$(5, 3)$
7	$\quad \neg\delta$	$\to$**E**$(1, 6)$
	$\quad \vdots$	
8	$\quad \delta$	$\to$**E**$(2, 4)$
9	$\quad \bot$	$\bot$**I**$(7, 8)$
10	$\neg\psi$	$\neg$**I**$(5$–$9)$

Solution for $(\psi \to \varphi) \land (\varphi \to \psi) \vdash (\varphi \land \psi) \lor (\neg\varphi \land \neg\psi)$ (I)

1	$(\psi \to \varphi) \land (\varphi \to \psi)$	
2	$\psi \to \varphi$	$\land\mathbf{E}_l(1)$
3	$\varphi \to \psi$	$\land\mathbf{E}_r(1)$
4	$\quad \neg((\varphi \land \psi) \lor (\neg\varphi \land \neg\psi))$	
5	$\quad\quad \psi$	
6	$\quad\quad \varphi$	$\to\mathbf{E}(2, 5)$
7	$\quad\quad \varphi \land \psi$	$\land\mathbf{I}(6, 5)$
8	$\quad\quad (\varphi \land \psi) \lor (\neg\varphi \land \neg\psi)$	$\lor\mathbf{I}_l(7)$
9	$\quad\quad \bot$	$\bot\mathbf{I}(4, 8)$

Solution for $(\psi \to \varphi) \wedge (\varphi \to \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$ (II)

	$\vdots$	
10	$\neg\psi$	$\neg$**I**$(5\text{–}9)$
11	φ	
12	ψ	$\to$**E**$(3, 11)$
13	$\varphi \wedge \psi$	$\wedge$**I**$(11, 12)$
14	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\vee$**I**$_l(13)$
15	$\bot$	$\bot$**I**$(4, 14)$
16	$\neg\varphi$	$\neg$**I**$(11\text{–}15)$

Solution for $(\psi \to \varphi) \wedge (\varphi \to \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$ (III)

	$\vdots$	
17	$\neg\varphi \wedge \neg\psi$	$\wedge\mathbf{I}(10, 16)$
18	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\vee\mathbf{I}_r(17)$
19	$\bot$	$\bot\mathbf{I}(4, 18)$
20	$\neg\neg((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$	$\neg\mathbf{I}(4\text{–}19)$
21	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\neg\mathbf{E}(20)$