Natural Deduction Rules for Propositional Logic

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Conjunction

The rules:

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\,\wedge\mathbf{I} \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\,\wedge\mathbf{E}_l \qquad\qquad \frac{\varphi \wedge \psi}{\psi}\,\wedge\mathbf{E}_r$$

Representation in Fitch's style:

$$\begin{array}{l|l}
 & \vdots \\
n & \varphi \\
 & \vdots \\
m & \psi \\
 & \vdots \\
 & \varphi \wedge \psi \qquad \wedge\mathbf{I}(n,m)
\end{array}
\qquad
\begin{array}{cl|ll}
 & \vdots & & \\
n & \varphi \wedge \psi & & \\
 & \vdots & & \\
m & \varphi & \wedge\mathbf{E}_l(n) & \\
\end{array}
\qquad
\begin{array}{cl|ll}
 & \vdots & & \\
n & \varphi \wedge \psi & & \\
 & \vdots & & \\
m & \psi & & \wedge\mathbf{E}_r(n)
\end{array}$$

Disjunction

The rules:

$$\frac{\varphi}{\varphi \vee \psi}\,\vee\mathbf{I}_l \qquad\qquad \frac{\psi}{\varphi \vee \psi}\,\vee\mathbf{I}_r \qquad\qquad \frac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \theta\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \theta\end{array}}{\theta}\,\vee\mathbf{E}$$

Representation in Fitch's style:

$$\begin{array}{r|ll} & \vdots & \\ n & \varphi & \\ & \vdots & \\ & \varphi \vee \psi & \vee\mathbf{I}_l(n) \end{array} \qquad \begin{array}{r|ll} & \vdots & \\ n & \psi & \\ & \vdots & \\ & \varphi \vee \psi & \vee\mathbf{I}_r(n) \end{array}$$

$$\begin{array}{r|ll} & \vdots & \\ n & \varphi \vee \psi & \\ & \vdots & \\ m_1 & \begin{array}{|l} \varphi \\ \hline \vdots \\ \theta \end{array} & \\ k_1 & & \\ & \vdots & \\ m_2 & \begin{array}{|l} \psi \\ \hline \vdots \\ \theta \end{array} & \\ k_2 & & \\ & \vdots & \\ & \theta & \vee\mathbf{E}(n, m_1{-}k_1, m_2{-}k_2) \end{array}$$

Negation

The rules:

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\neg\varphi}\,\neg\mathbf{I} \qquad\qquad \dfrac{\neg\neg\varphi}{\varphi}\,\neg\mathbf{E}$$

Representation in Fitch's style:

$$\begin{array}{ll|l} & \vdots & \\ n & \begin{array}{|l} \varphi \\ \hline \vdots \\ \bot \end{array} & \\ m & & \\ & \neg\varphi & \neg\mathbf{I}(n{-}m) \end{array} \qquad \begin{array}{ll|l} & \vdots & \\ n & \neg\neg\varphi & \\ & \vdots & \\ & \varphi & \neg\mathbf{E}(n) \end{array}$$

False

The rules:

$$\dfrac{\begin{array}{c}\varphi \\ \vdots \\ \neg\varphi\end{array}}{\bot}\,\bot\mathbf{I} \qquad \dfrac{\bot}{\varphi}\,\bot\mathbf{E}$$

Representation in Fitch's style:

$$\begin{array}{ll|l} & \vdots & \\ n & \varphi & \\ & \vdots & \\ m & \neg\varphi & \\ & \vdots & \\ & \bot & \bot\mathbf{I}(n,m) \end{array} \qquad \begin{array}{ll|l} & \vdots & \\ n & \bot & \\ & \vdots & \\ & \varphi & \bot\mathbf{E}(n) \end{array}$$

Implication

The rules:

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\,\to\mathbf{I} \qquad \dfrac{\varphi \to \psi \qquad \varphi}{\psi}\,\to\mathbf{E}$$

Representation in Fitch's style: