4. Average Time and Probabilistic Programs

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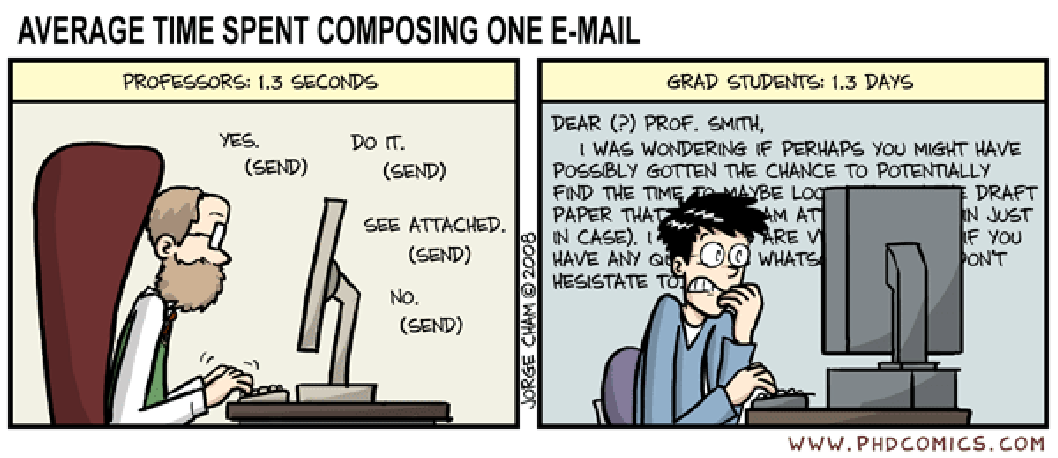
https://cister-labs.github.io/alg2324

Overview

- Measuring precisely performance of algorithms
- Measuring asymptotically performance of algorithms
- Analysing recursive functions
- Measuring precisely the average time of algorithms
- Possibly: sorting algorithms `bubbleSort`, `swapSort`, `insertionSort`, `mergeSort`, `quickSort`
- Next: analysis of sequences of operations (amortised analysis)



(from PhD comics: https://phdcomics.com/comics/archive.php?comicid=1047)

Recall goal

```
int count = 0;
for (int i=0; i<n; i++)
  if (v[i] == 0) count++
```

RAM

- worst-case: $T(n) = 5 + 5n$
- best-case: $T(n) = 5 + 4n$

#array-accesses + #count-increments

- worst-case: $T(n) = 2n$
- best-case: $T(n) = n$
- average-case: $\overline{T}(n) = n + \sum_{0 \leq r < n} P(v[r] = 0)$

Preliminaries: series

Recall arithmetic series

$$\sum_{i=1}^{n} i \;=\; 1+2+\ldots+n \;=\; \frac{n(n+1)}{2}$$

$$\sum_{i=a}^{b} i \;=\; a+(a+1)+\ldots+b \;=\; \frac{(a-b+1)(a+b)}{2}$$

Intuition

$$[\text{number of elements}] \times [\text{middle value}]$$

Recall geometric series I

$$\sum_{i=0}^{n} x^i \;=\; 1 + x + x^2 + \ldots + x^n \;=\; \frac{x^{n+1} - 1}{x - 1}$$

Proof

Let $S = \sum_{i=0}^{n} x^i$. Then:

$$S \times x \;=\; x + x^2 + \ldots + x^{n+1}$$

Hence we know $\left[(S \times x) - S \;=\; x^{n+1} - 1\right]$.

Simplifying we get $\left[S \;=\; \frac{x^{n+1}-1}{x-1}\right]$.

Recall geometric series II

$$\sum_{i=1}^{n} i \times x^{i-1} = x + (2 \times x^2) + \ldots + (n \times x^n) = \frac{n \times x^{n+1} - (n+1) \times x^n + 1}{(x-1)^2}$$

Proof

Recall $\left[S = \sum_{i=1}^{n} x^i = \frac{x^{n+1}-1}{x-1}\right]$. Derive both:

$$S' = (1 + x + x^2 + \ldots + x^n)' = 0 + 1 + 2x + \ldots + n \times x^{n-1} = \sum_{i=1}^{n} i \times x^{i-1}$$

$$\left(\frac{x^{n+1} - 1}{x - 1}\right)' = \frac{n \times x^{n+1} - (n+1) \times x^n + 1}{(x-1)^2}$$

Calculating average cases

Average case

The average time to execute an algorithm is given as the expected value for its execution, assuming that each run r has a cost c_r and a probability p_r.

Expected cost

$$\overline{T}(N) = \sum_{r} p_r \times c_r$$

Example: Linear search

```c
int lsearch(int x, int N, int v[])
{
  // pre: sorted array v
  int i;
  i =0;
  while ((i<N) && (v[i] < x))
    i ++;
  if ((i==N) || (v[i] != x))
    return (-1);
  else return i;
}
```

- Count array accesses
- Best case: $T(N) = 2$
- Worst case: $T(N) = N + 1$
- Average case: $\overline{T}(N) = \ldots$

Example: Linear search

```
int lsearch(int x, int N, int v[])
{
  // pre: sorted array v
  int i;
  i =0;
  while ((i<N) && (v[i] < x))
    i ++;
  if ((i==N) || (v[i] != x))
    return (-1);
  else return i;
}
```

- Count array accesses
- Best case: $T(N) = 2$
- Worst case: $T(N) = N + 1$
- Average case: $\overline{T}(N) = \dots$
  - assuming array with uniformly distributed values and a random `x`
  - same probability to do $0, 1, \dots, N-1$ cycle iterations
  - Hence: N different runs, each
    - probability: $1/N$
    - cost: $\#\text{cycles} + 1$

Example: Linear search

```
int lsearch(int x, int N, int v[])
{
  // pre: sorted array v
  int i;
  i =0;
  while ((i<N) && (v[i] < x))
    i ++;
  if ((i==N) || (v[i] != x))
    return (-1);
  else return i;
}
```

$$\overline{T}(N) = \sum_{i=1}^{N} \frac{1}{N} \times (i+1)$$

Example: Linear search

```
int lsearch(int x, int N, int v[])
{
  // pre: sorted array v
  int i;
  i =0;
  while ((i<N) && (v[i] < x))
    i ++;
  if ((i==N) || (v[i] != x))
    return (-1);
  else return i;
}
```

$$
\begin{aligned}
\overline{T}(N) &= \sum_{i=1}^{N} \frac{1}{N} \times (i+1) \\
&= \frac{1}{N} \times \sum_{i=1}^{N} (i+1) \\
&= \frac{1}{N} \times \sum_{i=2}^{N+1} i \\
&= \frac{1}{N} \times \frac{N \times (N+3)}{2} \\
&= \frac{N+3}{2}
\end{aligned}
$$

Binary search

```c
int bsearch(int x, int N, int v[])
{
  int i,s,m;
  i=0; s=N-1;
  while (i<s){
    m= (i+s)/2;
    if (v[m] == x) i = s = m;
    else if (v[m] > x) s = m-1;
    else i = m+1;
  }
  if ((i>s) || (v[i] != x))
    return (-1);
  else return i;
}
```

Ex. 4.1: Calculate best/worst/average cases

- Count array accesses / nr. cycles
- Best case: $T(N) = ?$
- Worst case: $T(N) = ?$
- Average case: $\overline{T}(N) = ?$

Binary search: Intuition for worst case

- Example: N=15, worst case
  - 1st cycle: check `v[N/2]` (7 remaining)
  - 2nd cycle: check `v[N/4]` (or `v[3N/4]` – 3 remaining)
  - 3rd cycle: check `v[N/8]` (or `v[3N/8]`... – 1 remaining)
  - after: check `v[N/16]` (or `v[3N/16]`...) if equal to `x`
- N=15, (3 cycles) $\rightarrow$ 4 “cycles”
- In general: c cycles for $2^c - 1$ elements
- ... i.e., $N = 2^c - 1 \quad \equiv \quad c = log_2(N + 1)$

Binary search: Intuition for average case

- In an array of size N, there are N+1 cases (finding at a given position, or not finding).
- Assume N+1 cases have equal probability (!)
- Example: N=15
  - 1 cycle: find at `v[N/2]` – prob. $\frac{1}{N+1}$
  - 2 cycles: find at `v[N/4]` or `v[3N/4]` – prob. $\frac{2}{N+1}$
  - 3 cycles: find at `v[N/8]` or (...) – prob. $\frac{4}{N+1}$
  - after: find (or not) at `v[N/16]` `(..)` – prob. $\frac{8}{N+1}$
- N=15, average cycles: $1 \times \frac{1}{N+1} + 2 \times \frac{2}{N+1} + 3 \times \frac{4}{N+1} + 4 \times \frac{8}{N+1}$
- In general: $1 \times \frac{1}{N+1} + \ldots + log_2(N+1) \times \frac{2^{log_2(N+1)-1}}{N+1}$
- ... i.e., $\overline{T}(N)) = \sum_{i=1}^{log_2(N+1)} i \times \frac{2^{i-1}}{N+1} = \ldots$

Two's complement

```c
void twoComplement(char b[], int N)
{
  int i = N-1;
  while (i>0 && !b[i])
    i --;
  i --;
  while (i >=0) {
    b[i] = !b[i];
    i--;
  }
}
```

Ex. 4.2: Calculate best/worst/average cases

- Count nr. *bit updates*
- Best case: $T(N) = ?$
- Worst case: $T(N) = ?$
- Average case: $\overline{T}(N) = ?$

Two's complement

```
void twoComplement(char b[], int N)
{
  int i = N-1;
  while (i>0 && !b[i])
    i --;
  i --;
  while (i >=0) {
    b[i] = !b[i];
    i--;
  }
}
```

Ex. 4.3: Calculate best/worst/average cases

- Count nr. *bit updates*
- Best case: $T(N) = ?$
- Worst case: $T(N) = ?$
- Average case: $\overline{T}(N) = ?$

twoComplement(0001) = 1111 – *1 vs -1*

twoComplement(0010) = 1110 – *2 vs -2*

twoComplement(0011) = 1101 – *3 vs -3*

twoComplement(01010000) = 10110000

Exercises

```c
int maxgrow(int v[], int N) {
  int r = 1, i = 0, m;
  while (i<N-1) {
    m = grow(v+i, N-i);
    if (m>r) r = m;
    i++;
  }
  return r;
}
```

```c
int grow(int v[], int N) {
  int i;
  for (i=1; i<N; i++)
    if (v[i] < v[i-1]) break;
  return i;
}
```

Ex. 4.4: How many comparison of array elements exist in the average case for `grow`? (assume `v[i]<v[i-1]` has 50% chances of succeeding)

Ex. 4.5: How many comparison of array elements exist in the average case for `maxgrow`?

Exercises @home

```
void iSort(int v[], int N){
  int i, j;
  for (i=1; i<N; i++)
    for (j=i; j>0 && v[j-1]>v[j];
           j--)
      swap(v,j,j-1);
}
```

Ex. 4.6: How many comparison of array elements exist in the average case?
(as before, assume `v[j-1]>v[j]` has 50% chances of succeeding)

Quicksort analysis

```c
int partition(int N, int v[]){
  int i, j=0;
  for (i=0; i<N-1; i++)
    if (v[i]<v[N-1])
      swap(v,i,j++);
  swap(v,N-1,j);
  return j ;
}
```

```c
void quickSort(int N, int v[]){
  int p;
  if (N>1) {
    p = partition(N, v);
    quickSort(v, p);
    quickSort(v+p+1, N-p-1);
  }
}
```

(See animation at `https://visualgo.net/en/sorting`)

Quicksort analysis

Partition

- Comparisons: $T_{\texttt{partition}}(N) = N - 1$ in any case
- Swaps: $T_{\texttt{partition}}(N) = N$ in the worst case, 1 in the best case

Quicksort (comparisons)

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N - 1 - p) & \text{if } N > 1\text{, where } 0 \leq p < N \end{cases}$$

Quicksort – worst case

Quicksort (comparisons) in general

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N-1-p) & \text{if } N > 1, \text{ where } 0 \leq p < N \end{cases}$$

Quicksort (comparisons) when $p = 0$

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(N-1) & \text{if } N > 1 \end{cases}$$

Quicksort – worst case

Quicksort (comparisons) in general

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N-1-p) & \text{if } N > 1, \text{ where } 0 \leq p < N \end{cases}$$

Quicksort (comparisons) when $p = 0$

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(N-1) & \text{if } N > 1 \end{cases}$$

$$\begin{aligned} T(N) &= (N-1) + (N-2) + \ldots + 2 + 1 \\ &= \sum_{i=1}^{N-1} i \;=\; \frac{N(N-1)}{2} \;=\; \Theta(N^2) \end{aligned}$$

Quicksort – best case

Quicksort (comparisons) in general

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N - 1 - p) & \text{if } N > 1\text{, where } 0 \leq p < N \end{cases}$$

Quicksort when $p = \frac{N-1}{2}$

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + 2T(\frac{N-1}{2}) & \text{if } N > 1 \end{cases}$$

Quicksort – best case

Quicksort (comparisons) in general

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N - 1 - p) & \text{if } N > 1\text{, where } 0 \leq p < N \end{cases}$$

Quicksort when $p = \frac{N-1}{2}$

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + 2T(\frac{N-1}{2}) & \text{if } N > 1 \end{cases}$$

$$\begin{aligned} T(N) &= ???(use\ recurrence\ trees) \\ &= \Theta(N \times log(N)) \end{aligned}$$

Quicksort – average case

Quicksort (comparisons) in general

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N-1-p) & \text{if } N > 1 \text{, where } 0 \leq p < N \end{cases}$$

Quicksort when p can be any with equal probability

$$\overline{T}(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + \sum_{p=0}^{N-1} \frac{1}{N}(\overline{T}(p) + \overline{T}(N-p-1)) & \text{if } N > 1 \end{cases}$$

Quicksort – average case

Quicksort (comparisons) in general

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + T(p) + T(N-1-p) & \text{if } N > 1\text{, where } 0 \leq p < N \end{cases}$$

Quicksort when p can be any with equal probability

$$\overline{T}(N) = \begin{cases} 0 & \text{if } N = 1 \\ N - 1 + \sum_{p=0}^{N-1} \frac{1}{N}(\overline{T}(p) + \overline{T}(N-p-1)) & \text{if } N > 1 \end{cases}$$

$$\sum_{p=0}^{N-1} \frac{1}{N}(\overline{T}(p) + \overline{T}(N-p-1)) \quad = \quad \frac{1}{N} \times \sum_{p=0}^{N-1} \overline{T}(p) + \frac{1}{N} \times \sum_{p=0}^{N-1} \overline{T}(N-p-1)$$

$$= \frac{1}{N} \times \sum_{p=0}^{N-1} \overline{T}(p) + \frac{1}{N} \times \sum_{p=0}^{N-1} \overline{T}(p) \quad = \quad \frac{2}{N} \times \sum_{p=0}^{N-1} \overline{T}(p)$$

Quicksort – average case (some math magic)

$$\overline{T}(N) = N-1+\sum_{p=0}^{N-1}\frac{1}{N}(\overline{T}(p)+\overline{T}(N-p-1)) = N-1+\frac{2}{N}\times\sum_{p=0}^{N-1}\overline{T}(p)$$

Multiplying by N

$$N\times\overline{T}(N) = N\times(N-1)+2\times\sum_{p=0}^{N-1}\overline{T}(p)$$

Applying for $N-1$

$$(N-1)\times\overline{T}(N-1) = (N-1)\times(N-2)+2\times\sum_{p=0}^{N-2}\overline{T}(p)$$

Subtracting each side

$$N\times\overline{T}(N)-(N-1)\times\overline{T}(N-1) = \\ N\times(N-1)+2\times\sum_{p=0}^{N-1}\overline{T}(p)-(N-1)\times(N-2)-2\times\sum_{p=0}^{N-2}\overline{T}(p)$$

Quicksort – average case (some math magic)

Subtracting each side

$$
\begin{aligned}
& N \times \overline{T}(N) - (N-1) \times \overline{T}(N-1) = \\
& \quad N \times (N-1) + 2 \times \textstyle\sum_{p=0}^{N-1} \overline{T}(p) - (N-1) \times (N-2) - 2 \times \textstyle\sum_{p=0}^{N-2} \overline{T}(p)
\end{aligned}
$$

Simplifying

$$
\begin{aligned}
\overline{T}(N) &= \left(\tfrac{2N-1}{N}\right) + \left(\tfrac{N+1}{N}\right) \times \overline{T}(N-1) \\
&= \ldots \\
&= \Theta(N \times log(N))
\end{aligned}
$$

Quicksort – average case (some math magic)

Subtracting each side

$$N \times \overline{T}(N) - (N-1) \times \overline{T}(N-1) = \\ N \times (N-1) + 2 \times \sum_{p=0}^{N-1} \overline{T}(p) - (N-1) \times (N-2) - 2 \times \sum_{p=0}^{N-2} \overline{T}(p)$$

Simplifying

$$\begin{aligned} \overline{T}(N) &= \left(\frac{2N-1}{N}\right) + \left(\frac{N+1}{N}\right) \times \overline{T}(N-1) \\ &= \ldots \\ &= \Theta(N \times log(N)) \end{aligned}$$

Randomised Quicksort – the version usually used – uses a random pivot when partitioning.

Randomised Algorithms

slides by Pedro Ribeiro, slides 4

pages 9-13

Randomized algorithms

We call an algorithm **randomized** if its behavior is determined not only by its input but also by values produced by a **random-number generator**

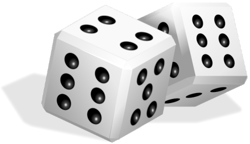
- Most programming environments offer a (deterministic) **pseudorandom-number generator**: it returns numbers that *"look"* statistically random
- We typically refer to the analysis of randomized algorithms by talking about the **expected cost** (ex: the **expected running time**)
- We can use **probabilistic analysis** to analyse randomized algorithms

Basics of Probabilistic Analysis

- Consider rolling **two dice** and observing the results.
- We call this an **experiment**.
- It has **36 possible outcomes**:

  1-1, 1-2, 1-3, 1-4, 1-5, 1-6, 2-1, 2-2, 2-3, ..., 6-4, 6-5, 6-6
- Each of these outcomes has probability $\mathbf{1/36}$ (assuming fair dice)
- What is the probability of the sum of dice being 7?

  Add the probabilities of all the outcomes satisfying this condition: 1-6, 2-5, 3-4, 4-3, 5-2, 6-1 (probability is $\mathbf{1/6}$)



Basics of Probabilistic Analysis

In the language of probability theory, this setting is characterized by a **sample space** S and a **probability measure** p.

- **Sample Space** is constituted by all possible outcomes, which are called **elementary events**
- In a **discrete probability distribution** (d.p.d.), the probability measure is a function $p(e)$ (or $Pr(e)$) over elementary events e such that:
  - $p(e) \geq 0$ for all $e \in S$
  - $\sum_{e \in S} p(e) = 1$
- An **event** is a subset of the sample space.
- For a d.p.d. the probability of an event is just the **sum** of the probabilities of its elementary events.

Basics of Probabilistic Analysis

- A **random variable** is a function from elementary events to integers or reals:

  Ex: let X_1 be a random variable representing result of first die and X_2 representing the second die.
  $X = X_1 + X_2$ would represent the sum of the two
  We could now ask: what is the probability that $X = 7$?

- One property of a random variable we care is **expectation**:

Expectation

For a discrete random variable X over sample space S, the expected value of X is:

$$\mathbf{E}[X] = \sum_{e \in S} Pr(e)X(e)$$

Basics of Probabilistic Analysis

- In **words**: the expectation of a random variable X is just its average value over S, where each elementary event e is weighted according to its probability.

  Ex: If we roll a single die, the expected value is 3.5
  (all six elementary events have equal probability).

- One possible rewrite of the previous equation, grouping elementary events:

Expectation (possible rewrite)

$$\mathbf{E}[X] = \sum_a Pr(X = a)a$$

Las Vegas vs. Monte Carlo

- QuickSort always returns a correct result (a sorted array) but its runtime is a random variable (with $\mathcal{O}(n \log n)$ in expectation)
- Some randomized algorithms are not guaranteed to be correct, but their runtime is fixed.

Las Vegas Algorithms

Randomized algorithms that always output the correct answer, and whose runtimes are random variables.

Monte Carlo Algorithms

Randomized algorithms that always terminate in a given time bound, but are correct with at least some (high) probability.