2. Algorithm Correctness

José Proença

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CISTER – U.Porto, Porto, Portugal

https://cister-labs.github.io/alg2324

Motivation

slides by Pedro Ribeiro, slides 1

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Correctness and Loop Invariants

Pedro Ribeiro

DCC/FCUP

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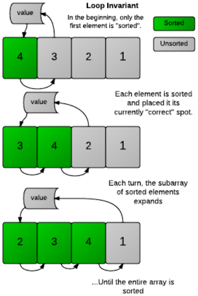
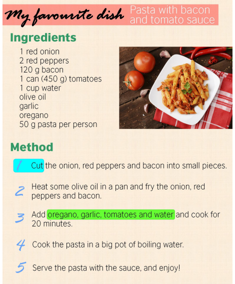
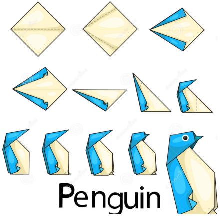


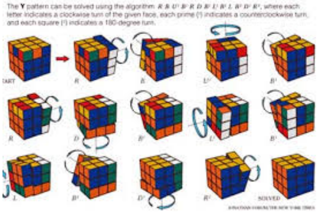
On Algorithms

What are algorithms? A set of **instructions** to solve a **problem**.

- The problem is the **motivation** for the algorithm
- The instructions need to be **executable**
- Typically, there are **different algorithms** for the same problem [how to choose?]
- **Representation**: description of the instructions that is understandable for the intended audience







On Algorithms

"Computer" Science version

- An algorithm is a **method** for solving a (computational) problem
- Algorithms are the **ideas** behind the programs and are independent from the programming language, the machine, ...
- A **problem** is characterized by the description of its **input** and **output**

A classical example:

Sorting Problem

Input: a sequence of $\langle a_1, a_2, \ldots, a_n\rangle$ of n numbers
Output: a permutation of the numbers $\langle a_1^{'}, a_2^{'}, \ldots, a_n^{'}\rangle$ such that $a_1^{'} \leq a_2^{'} \leq \ldots \leq a_n^{'}$

Example instance for the sorting problem

Input: 6 3 7 9 2 4
Output: 2 3 4 6 7 9

On Algorithms

What do we aim for?

- What **properties** do we want on an algorithm?

Correction

It has to solve correctly **all instances** of the problem

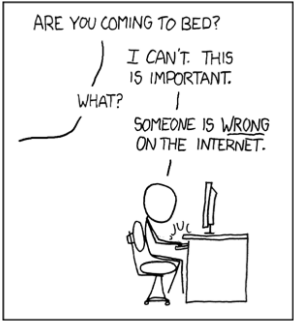
Efficiency

The performance (**time** and **memory**) has to be adequate

- This course is about **designing** correct and efficient algorithms and how to **prove** they meet the specifications

About correction

- In this lecture we will (mostly) worry about **correction**
  - Given an algorithm, it is not often obvious or trivial to know if it is **correct**, and even less so to **prove** this.
  - By learning how to reason about correctness, we also gain **insight** into what really makes an algorithm work



Specification

When is an algorithm correct?

Ex. 2.1: What do these functions do?

```
int fa (int x, int y){
  // pre: True
  ...
  // pos: (m == x || m == y) &&
  //      (m >= x && m >= y)
  return m;
}
```

```
int fb (int x, int y){
  // pre: x >= 0 && y >= 0
  ...
  // pos: x % r == 0 && y % r == 0
  return r;
}
```

```
int fc (int x, int y){
  // pre: x > 0 && y > 0
  ...
  // pos: r % x == 0 && r % y == 0
  return r;
}
```

```
int fd (int a[], int N){
  // pre: N>0
  ...
  // pos:
  //   (forall_{0<=i<N} x<=a[i]) &&
  //   (exists_{0<=i<N} x==a[i])
  return x;
}
```

When is an algorithm correct?

Ex. 2.2: Formulate pre- and post-conditions:

`int prod (int x, int y)` – product of two integers

`int gcd (int x, int y)` – greatest common divisor of 2 positive integers

`int sum (int v[], int N)` – sum of elements in an array

`int maxPOrd (int v[], int N)` – length of the longest sorted prefix of an array

`int isSorted (int v[], int N)` – tests if an array is sorted (growing)

Hoare triples

A triple $\{P\}S\{Q\}$ is a valid Hoare triple when

if [P *holds*] and [S *is executed*] then [Q *holds*]

Ex. 2.3: Find initial states that show these are not valid (and fix pre-cond.)

1. `{True} r=x+y; {r≥x}`
2. `{True} x=x+y; y=x-y; x=x-y; {x==y}`
3. `{True} x=x+y; y=x-y; x=x-y; {x≠y}`
4. `{True} if(x>y) r=x-y; else r=y-x; {r>0}`
5. `{True} while (x>0) {y=y+1; x=x-1;} {y>x}`

Partial correctness

Using rules for Hoare triples

$$\frac{P \Rightarrow Q[x\backslash E]}{\{P\}\ x := E\ \{Q\}} \qquad \frac{P \Rightarrow I \qquad \{I \land c\}\ S\ \{I\} \qquad (I \land \neg c) \Rightarrow Q}{\{P\}\ \texttt{while}\ c\ S\ \{Q\}}$$

1. **Initialisation:** $P \Rightarrow I$ *(P is the precondition right before the cycle)*
   Before the cycle the invariant holds.
2. **Maintenance:** $\{I \land c\}\ S\ \{I\}$ *(or $I \land c \Rightarrow I'$, where I' is the invariant after S)*
   Assuming the invariant holds before an iteration; it must be valid after it.
3. **Termination/Usefulness:** $(I \land \neg c) \Rightarrow Q$ *(simplify $I \land c$ until obtain Q)*
   After the cycle the post-condition holds.

slides by Pedro Ribeiro, slides 1

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Loops

- We will tackle one of the most fundamental (and most used) algorithmic patterns: a **loop** (e.g. `for` or `while` instructions)

Example loop: summing integers from 1 to n

$sum = 0$
$i = 1$
while $(i \leq n)$ {
  $sum = sum + i$
  $i = i + 1$
}

- We will talk about how to prove that a **loop** is correct
- We will show how this is also useful for **designing** new algorithms

Loop Invariants

Definition of Loop Invariant

A **condition** that is necessarily true immediately before (and immediately after) each iteration of a loop

Note that this says nothing about its truth or falsity part way through an iteration.

Instructions are for computers, invariants are for humans

- The loop program statements are **"operational"**, they are **"how to do"** instructions
- Invariants are **"assertional"**, capturing **"what it means"** descriptions

Anatomy of a loop

Consider a simple loop: **while (B)** { **S** }

- **Q**: precondition (assumptions at the beginning)
- **B**: the stop condition (defining when the loop end)
- **S**: the body of the loop (a set of statements)
- **R**: postcondition (what we want to be true at the end)

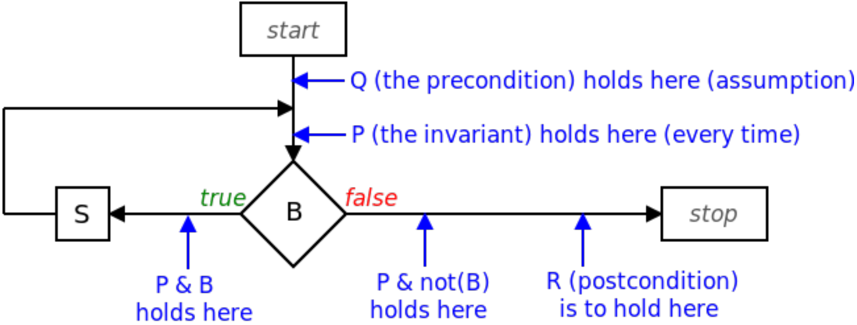
Example loop: summing integers from 1 to n

$sum = 0$
$i = 1$
while $(i \leq n)$ {
  $sum = sum + i$
  $i = i + 1$
}

- **Q**: $sum = 0$ and $i = 1$
- **B**: $i \leq N$
- **S**: $sum = sum + i$ followed by $i = i + 1$
- **R**: $sum = \sum_{i=1}^{n} i$

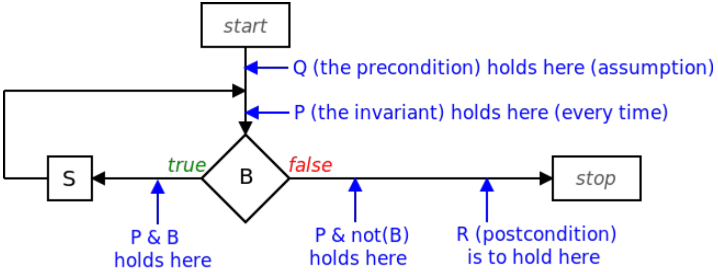
The invariant?

- **P**: an invariant (condition that holds at the start of each iteration)



- To be **useful**, the invariant P that we seek should be such that: $P \wedge not(B) \rightarrow R$
  - For the example sum loop, it could be: $sum = \sum_{i=1}^{i-1} i$

How to show that an invariant is really one?



- First, show that $Q \rightarrow P$
  (truth precondition Q guarantees truth of invariant P)
  - For the example sum loop: sum=0 which is $= \sum_{i=1}^{0} i$
- If $P \wedge B$, then after executing S, then P holds after executing S
  (the statements S of the loop guarantee that P is respected)
  - For the example sum loop: $\sum_{i=1}^{i-1} + i = \sum_{i=1}^{i}$

How to show that an invariant is really one?

Initialization

The invariant is true prior to the first iteration of the loop

Maintenance

If it is true before an iteration of the loop, it remains true before the next iteration

Termination

When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct

Exercises

```c
int mult1 (int x, int y){
  // pre: x>=0
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0){
    r = r+b;
    a = a-1;
  }
  // pos: r == x * y
  return r;
}
```

```c
int mult2 (int x, int y){
  // pre: x>=0
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0) {
    if (a%2 == 1) r = r+b;
    a=a/2;
    b=b*2;
  // pos: r == x * y
  return r;
}
```

Ex. 2.4: Check if *Initialization* and *Maintenance* holds for these formulae

$r == a * b$	$r \geq 0$	$b == 0$
$a \geq 0$	$a == x$	$a * b == x * y$
$b \geq 0$	$a \neq x$	$a * b + r == x * y$

Exercises

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    a=a/2;
    b=b*2;
  // pos: r == x * y
  return r;
}
```

Ex. 2.5: Find loop invariants to prove partial correctness

Some intuition – mult1(4,5)

```
1  int mult1 (int x, int y){
2    // pre: x>=0
3    int a, b, r;
4    a=x; b=y; r=0;
5    while (a>0){
6      r = r+b;
7      a = a-1;
8    }
9    // pos: r == x * y
10   return r;
11 }
```

line	x	y	a	b	r
4	4	5	4	5	0
6	4	5	4	5	5
7	4	5	3	5	5
6	4	5	3	5	10
7	4	5	2	5	10
6	4	5	2	5	15
7	4	5	1	5	15
6	4	5	1	5	20
7	4	5	0	5	20
10	4	5	0	5	20

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- x and y never change
- r grows proportionally as a shrinks
- guess:
  $I \triangleq a*y + r = x*y$

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- x and y never change
- r grows proportionally as a shrinks
- guess:
  $I \triangleq \texttt{a*y} + \texttt{r} = \texttt{x*y}$
- Need to show:

$$\texttt{x>=0} \Rightarrow I'$$

$$I \wedge \texttt{a>0} \Rightarrow I'$$

$$I \wedge \neg(\texttt{a>0}) \Rightarrow \texttt{r} = \texttt{x*y}$$

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- `x` and `y` never change
- `r` grows proportionally as `a` shrinks
- guess:
  $I \triangleq \texttt{a*y} + \texttt{r} = \texttt{x*y}$
- Need to show:

$$\texttt{x>=0} \Rightarrow I'$$

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$$I \wedge \neg(\texttt{a>0}) \Rightarrow \texttt{r} = \texttt{x*y}$$

- (Not all works – enrich invariant!)

More exercises

```
int serie(int n){
  // pre: n>=0
  int r=0, i=1;
  // inv: ??
  while (i!=n+1) {
    r = r+i; i = i+1;
  }
  // pos: r == n * (n+1) / 2;
  return r;
}
```

```
int mod(int x, int y) {
  // pre: x>=0 && y>0
  int r = x;
  while (y <= r) {
    r = r-y;
  }
  // pos: 0 <= r < y && exists_{q}
       x == q*y + r
  return r;
}
```

Ex. 2.5: Find loop invariants

Even more exercises (@home)

```
int minInd (int v[], int N) {
  // pre: N>0
  int i = 1, r = 0;
  // inv: ???
  while (i<N) {
    if (v[i] < v[r]) r = i;
    i = i+1; }
  // pos: 0 <= r < N && forall_{0 <= k < N} v[r] <= v[k]
return r; }

int minimum (int v[], int N) {
  // pre: N>0
  int i = 1, r = v[0];
  // inv: ???
  while (i!=N) {
     if (v[i] < r) r = v[i];
     i=i+1; }
  // pos: (forall_{0 <= k < N} r <= v[k]) &&
  //      (exists_{0 <= p < N} r == v[p])
  return r;
}

int sum (int v[], int N) {
  // pre: N>0
  int i = 0, r = 0;
  // inv: ???
  while (i!=N) {
    r = r + v[i]; i=i+1;
  }
  // pos: r == sum_{0 <= k < N} v[k]
  return r;
}
```

```
int sqr1 (int x) {
  // pre: x>=0
  int a = x, b = x, r = 0;
  // inv: ??
  while (a!=0) {
    if (a%2 != 0) r = r + b;
    a=a/2; b=b*2;
  }
  // pos: r == x^2
  return r;
  }

int sqr2 (int x){
  // pre: x>=0
  int r = 0, i = 0, p = 1;
  // inv: ??
  while (i<x) {
    i = i+1; r = r+p; p = p+2;
  }
  // pos: r == x^2
  return r;
}

int ssearch (int x, int a[], int N){
  // pre: N>0 &&
  //      forall_{0 < k < N-1} a[k-1]<=a[k]
  int p = -1, i = 0;
  // inv: ??
  while (p == -1 && i < N && x >= a[i]) {
    if (a[i] == x) p = i;
    i = i+1;
  }
  // pos: (p == -1 && forall_{0 <= k < N} a[k] != x) ||
  //      ( (0 <= p < N ) && x == a[p])
  return p;
}
```

Complete correctness

Partial/Complete correctness

Given $\{P\}\ S\ \{Q\}$

Partial correctness

if [P *holds*] and [S *is executed*] then [Q *holds*]

Complete correctness

if [P *holds*] and [S *is executed*] then [Q *holds*] AND S terminates

Partial/Complete correctness

Given $\{P\}\ S\ \{Q\}$

Partial correctness

if [P *holds*] and [S *is executed*] then [Q *holds*]

Complete correctness

if [P *holds*] and [S *is executed*] then [Q *holds*] AND S terminates

Enough to show the existence of a loop variant

Loop variant

Technique that measures the distance between the current state and the final state.

A loop variant V is an integer expression s.t.

- is positive in the beginning of each round ($c \wedge I \Rightarrow V > 0$)
- decreases in every round ($c \wedge I \Rightarrow V > V'$)

```
r=x;
q=0;
while (y <= r) {
  r = r-y;
  q = q+1;
}
```

- $V = r - y$ is not a good variant
- ...

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- $V = r - y + 1$ is a good variant

Loop variant

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```

- $V = r - y$ is not a good variant
- $V = r - y + 1$ is a good variant

  $y \leq r \Rightarrow V > 0$ at each round
  $V > V'$ after each round

Exercises

```
int sum(int v[], int N) {
  int i = 0, r = 0;
  while (i!=N) {
    // variant: ???
    r = r + v[i];
    i=i+1;
  }
  return r;
}
```

Ex. 2.6: Find variant above

Ex. 2.7: Find variants of the loops in previous exercises (when searching for invariants)